Chapter 5 Design of a Digital Sliding Mode Controller

In chapter 4 the linear controllers PID and RST are developed to regulate the Buck converter output voltage. Furthermore based on the PID and RST control laws, a Tri-mode controller is proposed to reduce power consumption inside. However SMPS converters are nonlinear system in nature due to their switching property. One of the most important features of the DC-DC converter is related to Variable Structure Systems (VSS) characterises. VSS are systems physically changed intentionally during time with respect to the structure control law. The instances at which the changing of the structure occurs are determined by the current state of the system. From this point of view SMPS represents a particular class of VSS since their structure is periodically changed by the action of controlled switches and diodes. Thus to enhance the dynamic performance of SMPS, nonlinear control methods, especially those nonlinear controllers with VSS are essential.

Sliding-Mode-Control (SMC), a form of the large group of VSS controller was theoretically introduced a few decades ago. SMC for VSS offers an alternative way to implement a control action which exploits the inherent variable structure nature of DC-DC converters [L1, P2, P3]. In particular the converter switches are driven as a function of the instantaneous values of the state variables in such a way so as to force the system trajectory to stay on a suitable selected surface on the phase space called the sliding surface. SMC has been improved suitable for DC-DC converters by several researchers in recent literatures [R7, R8, A5, S4, S5, S6].

However because of the SMC principles, the operation requires virtually infinite switching frequency that challenges the feasibility of SMC in low-power SMPS converters. In order to fix the very high and variable switching frequency, a fixed-frequency PWM-based SMC which derived from Hysteresis Modulator (HM) based SMC [B3, P4, M2] has been recently proposed in [S6, G1] but as an analog control for medium power SMPS applications where the switching frequency is in the range of hundreds kHz. Unfortunately analog PWM-based SMC is difficult to control high-frequency low-power SMPS. It is sensitive to analog component variations, and it can be mixed only with analog signals in the converter and not with signals inside a high-level digital power management unit. By contrast, digital DPWM-based SMC designed inside the controller is easier to implement, is insensitive to analog
component variations and offers more flexibility. Therefore, this chapter presents an original implementation of a DPWM-based SMC for a high-switching frequency SMPS.

5.1 Review of Sliding Mode Control

Before SMC application for digitally controlled SMPS, it is necessary to briefly introduce the theory and operation of SMC. This section covers the theoretical aspects of the SMC.

5.1.1 Sliding Mode Controller: An Ideal Controller in Theory

Sliding Mode Control (SMC) was introduced initially for the robust control of Variable Structure Systems (VSS) [C3, W2]. The basic principle of SMC is to employ a certain sliding surface as a reference path, such that the state variables trajectory can be directed towards the desired equilibrium. Theoretically such ideology of the SMC can be fully achieved only with the absolute compliances to certain conditions, namely the existence conditions, and the condition that the system operates at infinite switching frequency.

In such respect what is derived is an idealized controlled system, whereby no external disturbance or system uncertainties can affect the ideal control performance for zero regulation error and very fast dynamic response. These features fundamentally describe the expectancy of an ideally controlled system. Hence in a certain sense, the SMC is actually a type of ideal controller for the class of VSS.

5.1.2 Theory of Sliding Mode Controller

SMC theory has been applied to many nonlinear VSS to improve the systems performance. Variable Structure Controllers (VSC) act as a high-speed switched feedback control resulting in sliding mode. The purpose of the switching control law is to drive the nonlinear plant state trajectory onto a pre-specified (user-chosen) surface in the state space and to maintain the plant state trajectory on this surface for subsequent time. The surface is called a switching surface. When the plant state trajectory is “above” the surface, a feedback path has a given gain and another gain when the trajectory drops “below” the surface. This surface defines the rule for proper switching. The surface is also called a sliding surface (sliding manifold). Ideally, once intercepted, the switched control maintains the plant state trajectory on the surface for all subsequent time and the plant state trajectory slides along this surface until the origin. For example the surface \( S = 0 \) and the plant state trajectory of a SMC is shown in Fig.
5-1, where the SMC includes three state variables \( x_1, x_2 \) and \( x_3 \). The main idea of SMC is to bring and keep the error on a sliding surface such that the system is insensitive to the disturbances and parameter changes. Therefore the SMC is very robust.

The SMC design approach consists of two components:

<1> The first step is to build a sliding surface with the different forms of control objectives.

<2> The second step is to determine the existence of sliding motion and ensure the stability.

![Graphical representation of the plant state trajectory behaviour in SMC process](image)

Considering a general SISO (single-input-single-output) autonomous nonlinear system, the sliding-mode controlled system can be modelled by:

\[
\begin{align*}
\dot{x} &= f(x) + g(x)u \\
S(x) &= K^T x
\end{align*}
\]  

(5.1)

Where \( x \in \mathbb{R}^n \) is the variable vector of SMC, \( f \) and \( g \) are state vectors defined on \( \mathbb{R}^n \), \( S \) is the sliding surface, \( K^T = K_1, \ldots, K_n \) is the sliding gains parameters and \( u \) is the switch position which makes the system discontinuous:

\[
u = \begin{cases} 
  u^+, & \text{if } S(x) > 0 \\
  u^-, & \text{if } S(x) < 0 
\end{cases}
\]

(5.2)

where \( u^+ \) and \( u^- \) are the switching signals respectively in the range of \( S(x) > 0 \) and \( S(x) < 0 \).

In order that the sliding motion can exist, the state variable phase trajectory must be directed towards the sliding surface \( S(x) = 0 \) to obtain a stable solution of the system as shown in Fig. 5-1. Thus it is necessary to determine the sliding motion range for the SMC, i.e. to determine the existence conditions \([F1, J7, V2]\) and ensure the surface exist. This task can be performed using the Lyapunov’s second method \([J6]\), where the Lyapunov’s function \( V \) is generally defined as \( V = \frac{1}{2} S^2 x^2 \). The existence conditions must be satisfied:

\[
S(x) \dot{S}(x) < 0
\]

(5.3)

Thus the existence conditions of SMC can be written as:
\[
\begin{align*}
\dot{S}(x) &> 0 \text{ if } S(x) < 0 \\
\dot{S}(x) &< 0 \text{ if } S(x) > 0
\end{align*}
\] (5.4)

Substituting equation (5.1) and (5.2) into (5.4) then the existence conditions of SMC can be rewritten as:

\[
\begin{align*}
\dot{S}(x) &= K^T \left[ f(x) + g(x)u^+ \right] > 0 \text{ if } S \ x < 0 \\
\dot{S}(x) &= K^T \left[ f(x) + g(x)u^- \right] < 0 \text{ if } S \ x > 0
\end{align*}
\] (5.5)

In that way the existence conditions for the SMC are ensured.

5.2 Sliding Mode Control for DC-DC SMPS

DC-DC converters, a type of VSS, are non-linear in nature. Moreover the parameters of SMPS change with the load variation. One of the important features of the SMC in the VSS is its robustness, which makes the system insensitive to the parameters variation. From this point of view the SMC is particularly suitable for the application of SMPS converter.

5.2.1 Quasi-Sliding-Mode (QSM) Controller

In order to acquire high performance for VSS, the ideal SMC must be operated at high enough switching frequency on the sliding surface. Thus the nature of the sliding mode controller is to ideally operate at an infinite switching frequency such that the controlled variables can track a certain reference path to achieve the desired dynamic response and steady-state operation [V2]. This requirement for operation at infinite switching frequency, however, challenges the feasibility of applying SMC in DC-DC power converters. Hence for SMC to be applicable in power converters, their switching frequencies must be constricted within a practical range.

In order to limit the switching frequency, different methods such as Hysteresis-Mode, Constant-on-Time and Limited-Maximum-Frequency are proposed [B3]. Among these conventional limiting-frequency methods, the most popular way is the Hysteresis-Mode (HM-based) SMC that will be introduced in next sub-section.

Nevertheless this restriction of the sliding mode controller switching frequency transforms the sliding-mode controller into a type of quasi-sliding-mode (QSM) controller, which operates as an approximation of the ideal SMC. Since all SMC in practical power converters
are frequency-limited, they are, strictly speaking, a type of \textit{quasi-sliding-mode} (QSM) controllers.

\subsection*{5.2.2 Conventional HM-based SMC}

A review of the literature [L1, B3, M2, P4, V3] shows that most of the previously proposed SMC for switching power converters are \textit{Hysteresis-Modulation} (HM) based, which requires a bang–bang type controller to perform the switching control shown in Fig. 5-2.

![Hysteresis-base SMC](image)

**Fig. 5-2 Hysteresis Modulation-based SMC**

Since there are only two available choices "ON" (\(u = 1\)) and "OFF" (\(u = 0\)) for switch action in SMPS, then this method is easily accomplished as shown the equation (5.2):

\[
u = \begin{cases} 
1 = 'ON' & \text{if } S(x) > \sigma \\
0 = 'OFF' & \text{if } S(x) < -\sigma \\
\text{previous state} & \text{otherwise}
\end{cases} \quad (5.6)
\]

where \(\sigma\) is an arbitrarily small value around zero.

The introduction of a hysteresis band with the boundary conditions \(S(x) = \sigma\) and \(S(x) = -\sigma\) provides a form to limit the infinite high switching frequency. However due to the lack of systematic design methods and implementation criteria, the implementation of HM-based SMC for SMPS still relies on the trial-and-error tuning of the \(\sigma\) magnitude to achieve the desired switching frequency for a particular operating condition. The performance of HM-based SMC depends on the experience of designer and engineer.

\subsection*{5.2.3 The Requirement of Fixed-Frequency SMC}

Clearly from equation (5.6) that the infinite high switching frequency is limited by the hysteresis band value \(\sigma\), but the SMPS operation frequencies yet rely on the bang-bang magnitude \(\sigma\), i.e. HM-based SMC switching frequency is still variable, which inherits the typical disadvantages of having variable switching frequency operation and being highly control-sensitive to noise. Obviously, when switching frequency is variable, designing the
filters under a worst-case (lowest) frequency condition will result in oversized filters. Moreover, the variation of the switching frequency also deteriorates the regulation performance of the converters.

In order to keep the switching frequency fixed, two basic approaches have been proposed in the implementation of conventional HM-based SMC. One approach is to incorporate a constant ramp or timing function directly into the controller [P2, B3, L2]. However this method comes at an expense of additional hardware circuits, as well as deteriorated transient response in the system performance caused by the superposition of the ramp function upon the SMC switching function. Another approach is to include some forms of adaptive control into the HM-based SMC to contain the switching frequency variation [V3, S7]. However the architecture of this adaptive sliding mode controller is relatively more complex, and increase the implementation cost of the controller.

On the other hand, fixed switching frequency SMC can also be obtained by employing pulse-width modulation (PWM) instead of HM [S4, V4, S8]. In practice, this is similar to classical PWM control schemes in which the control signal is compared to the ramp waveform to generate a discrete gate pulse signal [D5]. The advantages of PWM-based SMC are that it does not need additional hardware circuitries since the switching function is performed by the PWM modulator, which can be implemented inside the digital controller. However in order to preserve the original sliding mode control laws, the practical implementation of PWM-based SMC is nontrivial, especially when both current and voltage state variables are involved. Hence this approach is not always implementable for some conventional HM-based SMC types.

5.2.4 PWM-Based SMC

Conventional HM-based SMC application is based on the equivalent control [V2, C3, W2], but PWM-based SMC application is based on the averaged duty control [S4, V4, S8, G1]. Because of the shortage of variable frequency operation in HM-based SMC, PWM-based SMC recently has been used to replace the HM-based SMC. However there is seldom theoretical analysis for the establishment of their relationship. In order to better understand how the PWM can replace the HM-based sliding mode control, it is necessary to briefly study equivalent control and averaged duty control.
A. Equivalent Control

As discussed previously, to achieve an ideal SMC operation, the system must be operated at an infinite switching frequency so that the state variables trajectory is oriented precisely on the sliding surface. However in the practical case of finite switching frequency, the trajectory will oscillate in the vicinity of the sliding surface while moving towards the origin (see Fig. 5-1 with three variables). It is possible to identify the movement of the trajectory as a composition of two isolate components: a fast-moving (high-frequency) component and a slow-moving (low-frequency) component shown in Fig. 5-3.

Fig. 5-3 High and low frequency components of the state trajectory on sliding surface where sliding surface $S = 0$ and $\dot{S} = 0$, $+ve$ and $-ve$ are the directions respectively below and above the surface $S$. It can be seen that the high-frequency component is actually a discontinuous trajectory that alternates between $+ve$ and $-ve$ direction, whereas the low-frequency component is actually a continuous trajectory that moves along the sliding plane. Since the movement of the trajectory is an effective consequence of the input switches action $u$, it is therefore possible to relate the corresponding low-frequency and high-frequency components of the trajectory to a low-frequency continuous switching action $u_{low}$, where $0 < u_{low} < 1$, and a high-frequency discontinuous switching action $u_{high}$, and that $u = u_{low} + u_{high}$. Under such assumptions, the switching action of $u_{high}$ produces only the high-frequency trajectory component, and the switching action of $u_{low}$ produces only the low-frequency trajectory component.

Because the high-frequency component is often filtered out by the SMPS converter filter, it is reasonable to consider the effect of the high-frequency discontinuous switching action $u_{high} = 0$, and only the low-frequency continuous switching action $u_{low}$ acts as the desired switching action $u$ that will produce a trajectory that is a near equivalence to an ideal SMC operation trajectory. This is the so-called equivalent control. The equivalent control signal, i.e., $u_{eq}$, is actually the low-frequency continuous switching action $u_{low}$ described above:

$$
\begin{cases}
  u = u_{low} \\ 
  u_{low} = u_{eq}
\end{cases} \Rightarrow u = u_{eq}
$$

(5.7)

Then the sliding-mode controlled system in equation (5.1) can be rewritten as
\[
\begin{align*}
\frac{dx}{dt} &= f(x) + g(x)u_{eq} \\
S(x) &= K^T x
\end{align*}
\]  

(5.8)

**B. Averaged Duty Control**

In conventional PWM control which is also known as the *averaged duty control*, the control input \( u \) is switched between ‘1’ and ‘0’ once per switching cycle for a fixed small duration \( \Delta \). The time instance where the switching occurs is determined by the sampled value of the state variables at the beginning of each switching cycle. Duty ratio \( d \) is then the fraction of the switching cycle in which the control holds the value 1. It is normally a smooth function of the state vector \( x \), and it is denoted by \( d(x) \) where \( 0 < d(x) < 1 \). Then for each switching cycle interval \( \Delta \) during the time \( \tau, \tau + \Delta \), the control input \( u \) can be written as

\[
u = \begin{cases} 
  u^+ = 1 & \text{if } \tau \leq t < \tau + \Delta \& x \Delta \\
  u^- = 0 & \text{if } \tau + \Delta \leq t < \tau + \Delta 
\end{cases}
\]  

(5.9)

It allows that a system \( x = f(x) + g(x)u \) can be expressed as

\[
x(\tau + \Delta) = x(\tau) + \int_{\tau}^{\tau+\Delta} x \, dt = x(\tau) + \int_{x}^{x+\Delta} f(x) + g(x) \, dt + \int_{x+\Delta}^{x+\Delta+\Delta} f(x) \, dt
\]  

(5.10)

The ideal average model of the PWM-controlled system response is obtained by allowing the duty ratio frequency to tend to infinity, i.e., \( \Delta \) to approach zero. Under such consideration, the above equation becomes

\[
\lim_{\Delta \to 0} \left[ x(\tau + \Delta) - x(\tau) \right] = \lim_{\Delta \to 0} \left[ \int_{\tau}^{\tau+\Delta} f(x) \, dt + \int_{x}^{x+\Delta} f(x) \, dt + \int_{x+\Delta}^{x+\Delta+\Delta} g(x) \, dt \right]
\]  

(5.11)

\[
= \lim_{\Delta \to 0} \left[ \int_{\tau}^{\tau+\Delta} f(x) \, dt + \int_{x}^{x+\Delta} g(x) \, dt \right]
\]

i.e.,

\[
\begin{align*}
\frac{dx}{dt} &= \dot{x} = f(x) + g(x) \quad d(x)
\end{align*}
\]  

(5.12)

which is referred as the averaged PWM-controlled system. Therefore, it is shown that as the duty frequency tends to infinity (\( \Delta \to 0 \)), the ideal average behaviour of the PWM-controlled system is represented by the smooth response of the system constituted by the duty ratio \( d \). It should also be noted that the duty ratio \( d \) replaces the discrete function \( u \) in the same manner as the equivalent control \( u_{eq} \) of the SMC scheme to obtain (5.8). Hence the relationship between *equivalent control* and *averaged duty control* can be established

\[
u_{eq} = d(x)
\]  

(5.13)
C. PWM-Based SMC Replace HM-Based SMC

Hence from the above review of equivalent control for HM-based SMC and averaged duty control for PWM-based SMC, it is very interesting that the PWM-based SMC can be used to replace the HM-based SMC. The PWM-based SMC can not only limit infinite high switching frequency but also can fix the variable switching frequency for SMPS application.

The analysis of equivalent control and averaged duty control has proven that their relationship can be established. First in sliding mode control, the discrete control input (gate signal) $u$ can be theoretically replaced by a smooth function known as the equivalent control signal $u_{eq}$ [V2, C3, W2, S7]. Second at high switching frequency, the equivalent control is effectively an averaged duty control signal $d$ [S4, V4, S8, G1]. Since averaged duty ratio is basically also a smooth analytic function of the discrete control pulses in PWM, we can obtain a PWM-based SMC by mapping the equivalent control function into the averaged duty control in the pulse-width modulator, i.e., $d = u_{eq}$. Finally the substitution of PWM-based SMC for HM-based SMC can be shown in Fig. 5-4.

![Fig. 5-4 PWM-based SMC is sued to replace HM-based SMC](image)

**5.3 Design of a DPWM-Based SMC**

Most of the existing SMC applied for SMPS converters are designed in analog control operating at low to medium frequency range (hundreds of kHz). This is not adequate to meet the requirements of small-size and high-frequency for today’s low-cost high-performance SMPS. These explain why the application of SMC in DC–DC converters has only been of academic/research interest but of little practical application.

Fortunately as the theoretical groundwork of SMC is fairly matured, PWM technique has been applied to SMC and digital CMOS technology has been rapidly developed, it is time to direct more research efforts towards developing practical high-frequency digital PWM-based SMC for SMPS converters. In this section we present a practical design of DPWM-based SMC application to a buck converter. Our focus in this design is the application of SMC to converter operating in continuous current mode (CCM), and the discontinuous current mode.
(DCM) is not discussed here. The system modeling, derivation of existence condition and selection of parameters for SMC are detailed in following sections.

### 5.3.1 System Modelling for Sliding-Mode Controller

The first step to the design of a SMC is to determine the state variables in terms of the desired the sliding model controller. Fig. 5-5 shows the schematic diagrams of the proposed PID-type SMC voltage controller for a buck converter. The sliding mode controller involves the output voltage error and its integral and differential portions. Unlike most existing SMC voltage controllers, besides the differential portion, it also takes into account an additional voltage error integral term to reduce the steady-state dc error of the output voltage.

![Schematic diagram of a PID-type SMC for a digitally controlled buck converter](image)

Where the output voltage $V_{out}$ is the SMC control objective, $K = K_1, K_2, K_3^T$ is the sliding parameter of SMC and the control variable $x$ can be expressed as:

$$
\dot{x} = \begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3 
\end{bmatrix} =
\begin{bmatrix}
    V_{ref} - V_{out} \\
    d\left(V_{ref} - V_{out}\right)/dt \\
    \int V_{ref} - V_{out} dt 
\end{bmatrix}
$$

(5.14)

where $V_{ref}$ represents the reference voltage, $x_1$, $x_2$ and $x_3$ are respectively the voltage error and its differential and integral portions. Extracting the time differentiation of equation (5.14) leads to

$$
\begin{align*}
    \dot{x} &= \begin{bmatrix}
        \dot{x}_1 \\
        \dot{x}_2 \\
        \dot{x}_3 
\end{bmatrix} =
\begin{bmatrix}
    d\left(V_{ref} - V_{out}\right)/dt \\
    \frac{d}{dt}\left(V_{ref} - V_{out}\right)/dt \\
    \frac{d}{dt}\int V_{ref} - V_{out} dt 
\end{bmatrix} =
\begin{bmatrix}
    x_2 \\
    x_3 \\
    x_1 
\end{bmatrix} 
\end{align*}
$$

(5.15)
For easy discussion, assuming that $V_{ref}$ is constant and capacitor ESR is zero, then

$$\dot{x}_1 = \frac{d}{dt} V_{ref} - V_{out} = -\frac{dV_{out}}{dt} = -\frac{dV}{dt} = -\frac{i}{C}$$

(5.16)

From equation (5.15) and (5.16), $x_2$ comes as:

$$\dot{x}_2 = \frac{d}{dt} x_1 = -\frac{d i_c}{dt} C$$

(5.17)

Considering the CCM operation,

$$i_c = i_L - i_{out}$$

(5.18)

Substituting equation (5.18) into (5.17) results in

$$\dot{x}_2 = -\frac{d}{dt} \left( i_L - i_{out} \right) C = -\frac{1}{C} \left( \frac{di_L}{dt} + \frac{di_{out}}{dt} \right)$$

(5.19)

Under the situation of CCM and averaged duty control,

$$\begin{cases}
V_L = \frac{di_L}{dt} \\
L = \frac{dt}
V_L = V_{in} \times u_{eq} - V_{out}
\end{cases}$$

(5.20)

Since

$$i_{out} = \frac{V_{out}}{R}$$

(5.21)

Then

$$\frac{di_{out}}{dt} = \frac{1}{R} \frac{dV_{out}}{dt}$$

(5.22)

Substituting equations (5.20) and (5.22) into (5.19), we can obtain

$$\dot{x}_2 = -\frac{1}{C} \left( \frac{V_{in} \times u_{eq} - V_{out}}{L} - \frac{1}{R} \frac{dV_{out}}{dt} \right)$$

(5.23)

Because of $x_1 = V_{ref} - V_{out}$, equation (5.23) can be rewritten as

$$\dot{x}_2 = -\frac{V_{in} \times u_{eq}}{LC} + \frac{V_{ref} - x_1}{LC} + \frac{1}{RC} \frac{dV_{out}}{dt}$$

(5.24)

Substituting equation (5.16) into (5.24), then

$$\dot{x}_2 = -\frac{V_{in} \times u_{eq}}{LC} + \frac{V_{ref} - x_1}{LC} - \frac{x_1}{RC}$$

(5.25)

Equation (5.17) finally leads to:

$$\dot{x}_2 = \frac{V_{in} \times u_{eq} + V_{ref}}{LC} - \frac{x_1}{LC} - \frac{x_2}{RC}$$

(5.26)

Substituting equation (5.26) into equation (5.15), then

$$\dot{x} = \begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
-1 & -1 & 0 \\
1 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix} + \begin{bmatrix}
0 \\
-V_{in} \times u_{eq} + \frac{V_{ref}}{LC} \\
0 \\
\end{bmatrix}$$

(5.27)
Equation (5.27) can be rewritten in a standard form of state-space description:

\[
x = Ax + Bu_{eq} + H
\]  

(5.28)

Confronting with (5.8), it is clear that

\[
\begin{align*}
\dot{x} &= f(x) + g(x) u_{eq} \\
 f(x) &= Ax + H, \quad g(x) = B \\
 S\ x &= K^T x = K_1 x_1 + K_2 x_2 + K_3 x_3 \\
 \dot{S}\ x &= K^T \dot{x} = K^T \left[ Ax + Bu_{eq} + H \right]
\end{align*}
\]

(5.29)

### 5.3.2 Derivation of DPWM-Based SMC

As mentioned previously, since the relationship between equivalent control and averaged duty control has been established and the equivalent control signal \( u_{eq} \) has been proven equivalent to PWM duty ratio \( d \) for SMC, then the derivation of the DPWM-based SMC can be achieved by the equivalent control on the sliding surface \( S \). Combining the preceding equations (5.4), (5.27) and (5.28) with (5.29), we can derivate the equivalent control input signal \( u_{eq} \) using the invariance conditions by setting \( \dot{S}\ x = 0 \), i.e.,

\[
\dot{S}\ x = K^T \dot{x} = K^T \left[ Ax + Bu_{eq} + H \right] = 0
\]  

(5.30)

Now solving for equivalent control function (5.30) yields

\[
u_{eq} = -K^T B^{-1} K^T Ax + H = \frac{1}{V_{in}} \left[ LC \left( \frac{K_1}{K_2} - \frac{1}{RC} \right) x_2 + LC \left( \frac{K_3}{K_2} - \frac{1}{LC} \right) x_1 + V_{ref} \right]
\]  

(5.31)

Substituting \( x_1 = V_{ref} - V_{out} \) and \( x_2 = x_1 = -\frac{dV_{out}}{dt} \) into equation (5.31), then

\[
u_{eq} = \frac{1}{V_{in}} \left[ V_{ref} - LC \left( \frac{K_1}{K_2} - \frac{1}{RC} \right) \frac{dV_{out}}{dt} + LC \left( \frac{K_3}{K_2} - \frac{1}{LC} \right) V_{ref} - V_{out} \right]
\]  

(5.32)

where \( u_{eq} \) is continuous and equals to PWM duty ratio \( d \), and \( 0 < u_{eq} = d < 1 \), parameters \( K_1/K_2 \) and \( K_3/K_2 \) are to be determined which corresponds to the desired SMC dynamics that will be discussed in next section.

A close inspection of equation (5.32) reveals that the control signal \( u_{eq} \) involves the time differentiation of output voltage, \( dV_{out}/dt = i_c/C \), which results in the need of measurements (such as current sensor or current-to-voltage sampling circuit) for capacitor current \( i_c \) and thus increase the size and cost of the digital controller. Since the buck converter has the output voltage feedback \( V_{out} \) and the SMC controller is implemented in full digital form, it is feasible to design a software observer or a mathematic function for \( V_{out} \) to eliminate the need for hardware measurement. Unfortunately sophisticated software observer needs supplement
The observer design issue is not discussed here. A simple numeric derivation is adopted to calculate the rate of output voltage changing in the FPGA implementation:

$$\frac{dV_{out}}{dt} = \frac{V_{out} n - V_{out} n-1}{T_s} = \dot{V}_n$$  \hspace{1cm} (5.33)

where $V_{out} n$ and $V_{out} n-1$ are the output voltage in $n^{th}$ and $(n-1)^{th}$ cycle respectively, $T_s$ is PWM switching period. Link equations (5.32) and (5.33), we can obtain the duty ratio of PWM-based SMC controller

$$u_{eq} = \frac{1}{V_{in}} \left[ V_{ref} - LC \left( \frac{K_1}{K_2} - \frac{1}{RC} \right) \dot{V}_s + LC \left( \frac{K_3}{K_2} - \frac{1}{LC} \right) V_{ref} - V_{out} \right]$$  \hspace{1cm} (5.34)

**5.3.3 Determination of SMC Parameters**

Equation (5.34) gives the complete information for the duty ratio signal $u_{eq}$ of DPWM-based SMC, where two sliding parameters $K_1/K_2$ and $K_3/K_2$ still need to be determined corresponding to the desired dynamics. For this purpose we employ the Ackermann’s Formula [J7] to design the SMC also called as PID-type SMC to select the sliding parameters in this case. The selection of sliding parameters is based on the desired second-order dynamic properties. In this way, the sliding motion ensures that the state trajectory of the system under SMC operation will always reach a stable equilibrium point. As stated in Equations (5.4) and (5.14), the surface of the PID-type SMC at the stable equilibrium point, $S x = 0$ can be detailed as:

$$S x = K^T x = K_1 x_1 + K_2 x_2 + K_3 x_3 = K_1 x_1 + K_2 \frac{dx_1}{dt} + K_3 \int x_1 dt = 0$$  \hspace{1cm} (5.35)

According to Ackermann’s Formula [J7] for a standard second-order system dynamics, the equation (5.35) can be transformed into

$$\frac{d^2 x_1}{dt^2} + \frac{K_1}{K_2} \frac{dx_1}{dt} + \frac{K_3}{K_2} x_1 = 0$$  \hspace{1cm} (5.36)

Equation (5.43) can be rewritten as

$$\ddot{x}_1 + 2\zeta \omega_n x_1 + \omega_n^2 x_1 = 0$$  \hspace{1cm} (5.37)

where $\omega_n = \sqrt{K_3 / K_2}$ is the undamped natural frequency and $\zeta = K_1 / 2\sqrt{K_2 K_3}$ is the damping ratio. Thus the sliding parameters now depend on the desired dynamic performance that can be chosen by user.
5.3.4 Derivation of Existence Conditions

To achieve the design specifications, the SMC must maintain the variables state trajectory on the surface for all subsequent time and the state trajectory slides along this surface. Thus it is necessary to determine the sliding motion range for the SMC, i.e., to ensure the surface exist and to determine the existence conditions for sliding surface. As stated previously, this task is performed using the Lyapunov’s second method [16]. Recalling of the existence conditions in equation (5.5), the existence conditions must be satisfied:

\[
\begin{align*}
\dot{S}(x) &= K^T \left[ f(x) + g(x)u^+ \right] > 0 \text{ if } S \ x < 0 \\
\dot{S}(x) &= K^T \left[ f(x) + g(x)u^- \right] < 0 \text{ if } S \ x > 0
\end{align*}
\] (5.38)

Substituting equation (5.2) and (5.29) into (5.30),

\[
\begin{align*}
\dot{S}(x)_{s \rightarrow 0^+} &= K^T \left[ Ax + Bu^+ + D \right] > 0 \text{ if } S \ x \rightarrow 0^+ \\
\dot{S}(x)_{s \rightarrow 0^-} &= K^T \left[ Ax + Bu^- + D \right] < 0 \text{ if } S \ x \rightarrow 0^-
\end{align*}
\] (5.39)

Detailing equation (5.39) leads to

- Case 1: \( S \rightarrow 0^+ \), \( u^+ = 1 \), then \( \dot{S} < 0 \):
  
  Equation (5.39) can be detailed:
  \[
  \dot{S} \ x_{s \rightarrow 0^+} = \left( K_1 - \frac{K_2}{RC} \right) x_2 + \left( K_3 - \frac{K_2}{LC} \right) x_1 + \frac{K_2}{LC} V_{ref} - V_{in} < 0
  \] (5.40)

- Case 2: \( S \rightarrow 0^- \), \( u^- = 0 \), then \( \dot{S} > 0 \):
  
  Equation (5.39) can be detailed:
  \[
  \dot{S} \ x_{s \rightarrow 0^-} = \left( K_1 - \frac{K_2}{RC} \right) x_2 + \left( K_3 - \frac{K_2}{LC} \right) x_1 + \frac{K_2}{LC} V_{ref} > 0
  \] (5.41)

It can be seen that \( x_1 \), \( x_2 \) and \( S \ x \) construct the sliding surface in a 3-D space, which is the existence range of sliding motion for SMC. In order to simplify the analysis of the existence, the 3-D space can be mapped onto the 2-phase-plane \( (x_1 x_2) \). Assuming \( K_2 \) is positive, the existence range of surface for the PID-type SMC, equation (5.40) and (5.41), can be represented with the two lines in \( x_1 x_2 \) phase-plane:

\[
\begin{align*}
\lambda_1 \ x &= \left( K_1 - \frac{K_2}{RC} \right) x_2 + \left( K_3 - \frac{1}{LC} \right) x_1 + \frac{V_{ref} - V_{in}}{LC} < 0 \\
\lambda_2 \ x &= \left( K_1 - \frac{K_2}{RC} \right) x_2 + \left( K_3 - \frac{1}{LC} \right) x_1 + \frac{V_{ref}}{LC} > 0
\end{align*}
\] (5.42)

Thus the two lines \( \lambda_1 \ x = 0 \) and \( \lambda_2 \ x = 0 \) respectively determine the two boundaries of sliding surface existence range. From equation (5.42), it can be seen the two lines has the
same slope in the $x_1x_2$ phase plane, where line \( \lambda_1 x = 0 \) passes through the point \( A \): \[
\left[ \frac{V_{in} - V_{ref}}{K_1/K_2 - 1/LC} \right]/LC , 0 \] and point \( B \): \[
\left[ \frac{V_{in} - V_{ref}}{K_1/K_2 - 1/RC} \right]/LC , 0 \] line \( \lambda_2 x = 0 \) passes through the points point \( C \): \[
\left[ -\frac{V_{ref}}{K_3/K_2 - 1/LC} \right]/LC , 0 \] and point \( D \): \[
\left[ -\frac{V_{ref}}{K_3/K_2 - 1/RC} \right]/LC , 0 \] and the axis \( x_1 = V_{ref} - V_{out} \) is limited with the minimum value point \( E \): \( V_{ref} - V_{in} , 0 \) and the maximum value point \( V_{ref} , 0 \). For different values of the parameters \( K_1/K_2 \) and \( K_3/K_2 \), the slope of the two parallel lines is variable. Therefore the existence conditions can be illustrated in two situations: (1) positive slope region and (2) negative slope region for each line. As stated above, the sliding parameters \( K_1/K_2 \) and \( K_3/K_2 \) are variable with different dynamical response frequency \( f_{cr} \), thus the points \( A, B, C \) and \( D \) will be located at axis in positive or negative region. The regions of existence for SMC is shown in Fig. 5-6 with four possibilities: (a) \( K_1/K_2 < 1/ RC \) and \( K_3/K_2 < 1/ LC \), (b) \( K_1/K_2 > 1/ RC \) and \( K_3/K_2 < 1/ LC \), (c) \( K_1/K_2 < 1/ RC \) and \( K_3/K_2 > 1/ LC \), and (d) \( K_1/K_2 > 1/ RC \) and \( K_3/K_2 > 1/ LC \), where (a) and (d) belong to situation (2) with negative slope, (b) and (c) belong to situation (1) with positive slope.

According to equation (5.40), when the buck converter operates at 4 MHz, the parameters are \( K_1/K_2 < 1/ RC \) and \( K_3/K_2 > 1/ LC \). Thus the existence region for the PID-type SMC in this case can be represented in Fig. 5-6 (c). The existence conditions show the region of existence of the SMC, which provides a range of employable sliding area that will ensure the state trajectory keep sliding along the surface \( S(x) = 0 \) until reaching the stable operation at the origin \( O \).

![Fig. 5-6 region of existence for the sliding mode mapped in the phase plane with four possibilities (a, b, c, d)](image-url)
5.3.5 Stability for SMC

Besides the existence conditions, the sliding mode controller should also comply with the stability conditions. The stability is an important item which is to ensure that the sliding surface will direct the state trajectory toward the stable equilibrium points in existence regions.

In this work the analysis of dynamic response and stability for the PID-type SMC can be started the surface $S \ x = 0$ and $\dot{S} \ x = 0$. Since the state variables $x = x_1, x_2, x_3$ are in phase canonical form, the $S \ x = 0$ can be rewritten in Laplace form as:

$$K_1X + K_2sX + K_3X \ s = 0 \Rightarrow s^2 + \frac{K_3}{K_2} s + \frac{K_1}{K_2} = 0 \quad (5.43)$$

Firstly using the Routh’s stability criterion to this second order linear polynomial $\Delta$ to determine the stability conditions,

$$\Delta = \begin{vmatrix} s^2 & 1 & \frac{K_3}{K_2} \\ s^1 & \frac{K_1}{K_2} & 0 \\ s^0 & \frac{K_1 K_3}{K_2^2} & \end{vmatrix} \quad (5.44)$$

The condition for the stability must meet $\Delta > 0$, which means all the coefficients $K_1, K_2, K_3$ must be with the same sign (positive or negative), i.e., $K_{1,2,3} > 0$ or $K_{1,2,3} < 0$. This can ensure all roots have negative real parts.

Secondly Extracting the time differential, the $\dot{S} \ x = 0$ can be rearranged into a stand second-order system form:

$$s^2 + 2\zeta \omega_n s + \omega_n^2 = 0 \quad (5.45)$$

which is identical to equation (5.37), where the values of damping ratio $\zeta$, undamped natural frequency $\omega_n$ and two eigenvalues $s_{1,2}$ are

$$\omega_n = \sqrt{\frac{K_3}{K_2}}, \quad \zeta = \frac{K_1}{2\sqrt{K_3 K_2}} \quad (5.46)$$

$$s_{1,2} = -\zeta \omega_n \pm j\omega_n \sqrt{\zeta^2 - 1}$$

Since $\zeta > 0$ then $K_1$ must be positive. Recalling that $K_1, K_2, K_3$ must be with the same sign, thus finally all the coefficients should be positive for stable operation.

5.3.6 Matlab Simulation of SMC for a Buck Converter

The time domain behaviour of the proposed digital DPWM-based SMC is verified on a buck converter using Matlab/ Simulink shown in Fig. 5-6, where the buck converter circuit elements are: $L = 4.7 \mu H$, $C = 22 \mu F$, $R = 5\Omega$, $V_{in} = 3.0V$, $V_{out} = 1.5V$ and switching frequency $f_s = 4MHz$. 

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To comply with the design equations regarding with the stability and the transient response, the choice for system dynamic performance should be considered with the trade-off between $\omega_n$ and $\zeta$, where large of simulations have been carried out. Finally we set the bandwidth of the SMC response $f_w$ at one-fifteenth of the switching frequency $f_s$, i.e., $f_w = \frac{15 \times \omega_n}{2 \pi}$, and choose the damping ratio $\zeta = 1$. Thus the sliding parameters are determined as:

$$K_1/K_2 = 4\pi f_s/15 \quad \text{and} \quad K_1/K_2 = 4\pi^2 f_s^2/15^2$$

(5.47)

The modelling of the buck converter is modelled in a hybrid model [S12] using Matlab s-function and all the calculations are computed in fixed-point computation with 13-bit fraction (see section 4.2.3). To meet the condition of non-limit cycle $N_{PWM} > N_{ADC}$ [A7], the analog-to-digital model has a 10-bit resolution and DPWM model has an 11-bit resolution. The simulation results and their analysis are shown as below.

Initially the output voltage $V_{out}$ follows the reference $V_{ref}$ by a slope function until steady state. Then the load suddenly changes from 0.3A to 0.46A (R: 5Ω→3.3Ω), Fig. 5-7 shows the transient response of the SMC at switching frequency 4MHz: output voltage (a), PWM duty ratio (b) inductance current (c). It can be seen that the dynamic response of SMC is satisfying, where the transient response is very fast and the undershot is small.

To illustrate the details of sliding trajectory in the SMC operation, Fig. 5-8 shows the sliding trajectory in $x_1-x_2-S(x)$ 3-D space (a) and in $x_1-x_2$ two phase plane (b). During the operation of the slope function, the output voltage $V_{out}$ always tracks the $V_{ref}$ and the sliding surface $S$ directs toward the equilibrium point until steady state $V_{out} = V_{ref}$. In steady state the sliding trajectory is on the stable surface near the origin, and in dynamic state the equilibrium is broken and the sliding trajectory will directs toward the new equilibrium point again. Consequently as shown in Fig. 5-8 the sliding trajectory changes twice in two circles. In order to clearly explain the change of the sliding trajectory during the SMC operation, the sliding surface $S$ is also illustrated in the time-domain. Thus the Fig. 5-9 shows the sliding trajectory in $t-x_1-S(x)$ 3-D space (a) and in $t-S(x)$ two phase plane (b). It can be seen when load changes the sliding trajectory can quickly direct to the stable equilibrium.
Fig. 5.7 Dynamic response of SMC when load changes from 0.3A to 0.46A (R: 5Ω→3.3Ω): output voltage $V_{out}$ (a), inductance current $I_L$ (b) and PWM duty ratio $d$ (c) at switching frequency 4MHz.
Fig. 5.8 SMC Sliding trajectory in $x_1$-$x_2$-$S(x)$ 3-D space (a) and mapped in $x_1$-$x_2$ two-phase plane (b) at switching frequency 4MHz
Initially, SMC operates in a slope function, steady state. Dynamic state steady state

Fig. 5-9 SMC Sliding trajectory in time-domain: $t-x_1-S(x)$ 3-D space (a) and mapped in $t-S(x)$ two-phase plane (b) at 4 MHz switching frequency
Fig. 5-10 Dynamic response of SMC when load changes from 15mA to 0.46A (R: 100Ω→3.3Ω): output voltage $V_{out}$ (a), inductance current $I_L$ (b) and PWM duty ratio $d$ (c) at switching frequency 4MHz.
Initially, SMC operates in a slope function, steady state, dynamic state, and steady state.

Fig. 5-11 Comparison results of $V_{\text{out}}$, duty ratio $d$ and sliding trajectory $S$ for SMC when load changes from 15mA to 0.46A (R: 100Ω→3.3Ω) at 1MHz, 2MHz and 4MHz respectively.
In order to validate the disturbance rejection in larger range of variation, Fig. 5-10 shows the dynamic results when the load changes from 15mA to 0.46A (R: 100Ω→3.3Ω). It can be seen that the SMC can well regulate the output voltage even in large range of load variation.

In order to compare the SMC operation performance at different switching frequency, Fig. 5-11 shows the comparison results of $V_{out}$, duty ratio $d$ and sliding trajectory $S$ for SMC when load changes from 15mA to 0.46A (R: 100Ω→3.3Ω) at 1MHz, 2MHz and 4MHz respectively. It can be seen that the dynamic response of SMC is better with increasing switching frequency. As stated previously that higher switching frequency performs in higher dynamic performance, and an infinite frequency directs toward an ideal SMC.

5.4 Summary

In this chapter a nonlinear DPWM-based sliding mode controller which is derived from the conventional analog HM-based SMC, is proposed for the digitally controlled high-frequency low-power SMPS application. Firstly a brief review of sliding mode control is given in details, which contains the fundamental theory and existence conditions for SMC. Subsequently the SMC is introduced to the SMPS application domain. The ideal controller in theory is redefined to meet practical limitations. Hence the Quasi-Sliding-Mode (QSM) control is introduced to reduce the infinite high switching. For the sake of completeness, we present the conventional HM-based SMC and reveal its shortage in practical design. To meet the requirement of fixed-frequency, a PWM modulator is introduced to solve the problem of frequency variation. The relationship between the equivalent control and averaged duty control is derived, so that the PWM-based SMC is adopted to replace HM-based SMC. Most of the conventional HM-based SMC and recent PWM-based SMC are designed as analog controllers in low to medium power level, where the switching frequency and control performance are not adequate for todays SMPS. Thus we present an original implementation of a DPWM-based SMC for a high-switching low-power frequency SMPS. An example design of DPWM-based SMC for a buck converter is detailed. The practical design involves system modelling, derivation of DPWM-based SMC, selection of sliding parameters, derivation of existence and stability analysis. The time domain behaviour of the proposed digital DPWM-based SMC is verified on a buck converter using Matlab/ simulink. The simulation results verify the performance of proposed SMC. The FPGA implementation will be presented in chapter 6.